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On families of convex polytopes with constant metric dimension[☆]

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ABSTRACT

Let G be a connected graph and $d(x, y)$ be the distance between the vertices x and y. A subset of vertices $W = \{w_1, w_2, \dots, w_k\}$ is called a resolving set for G if for every two distinct vertices $x, y \in V(G)$, there is a vertex $w_i \in W$ such that $d(x, w_i) \neq d(y, w_i)$. A resolving set containing a minimum number of vertices is called a metric basis for G and the number of vertices in a metric basis is its metric dimension $dim(G)$. A family $\mathcal{G}$ of connected graphs is a family with constant metric dimension if $dim(G)$ is finite and does not depend upon the choice of G in $\mathcal{G}$.

In this paper, we study the metric dimension of some classes of convex polytopes which are obtained by the combinations of two different graph of convex polytopes. It is shown that these classes of convex polytopes have the constant metric dimension and only three vertices chosen appropriately suffice to resolve all the vertices of these classes of convex polytopes.

1. Notation and preliminary results

A basic problem in chemistry is to provide mathematical representations for a set of chemical compounds in a way that gives distinct representations to distinct compounds. As described in [1], the structure of a chemical compound can be represented by a labeled graph whose vertex and edge labels specify the atom and bond types, respectively. Thus, a graph-theoretic interpretation of this problem is to provide representations for the vertices of a graph in such a way that distinct vertices have distinct representations. This is the subject of papers [1–5].

In a connected graph, the *distance* $d(u, v)$ between two vertices $u, v \in V(G)$ is the length of a shortest path between them. Let $W = \{w_1, w_2, \dots, w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The *representation* $r(v|W)$ of v with respect to W is the k-tuple $(d(v, w_1), d(v, w_2), d(v, w_3), \dots, d(v, w_k))$. W is called a *resolving set* [1] or *locating set* [4] if every vertex of G is uniquely identified by its distances from the vertices of W, or equivalently, if distinct vertices of G have distinct representations with respect to W. A resolving set of minimum cardinality is called a *basis* for G and this cardinality is the *metric dimension* of G, denoted by $dim(G)$ [2]. The concepts of resolving set and metric basis have previously appeared in the literature (see [1–16]).

For a given ordered set of vertices $W = \{w_1, w_2, \dots, w_k\}$ of a graph G, the ith component of $r(v|W)$ is 0 if and only if $v = w_i$. Thus, to show that W is a resolving set it suffices to verify that $r(x|W) \neq r(y|W)$ for each pair of distinct vertices $x, y \in V(G) \setminus W$.

A useful property in finding $dim(G)$ is the following lemma:

Lemma 1 ([16]). *Let W be a resolving set for a connected graph G and $u, v \in V(G)$. If $d(u, w) = d(v, w)$ for all vertices $w \in V(G) \setminus \{u, v\}$, then $\{u, v\} \cap W \neq \emptyset$.*

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Motivated by the problem of uniquely determining the location of an intruder in a network, the concept of metric dimension was introduced by Slater in [4,5] and studied independently by Harary and Melter in [17]. Applications of this invariant to the navigation of robots in networks are discussed in [14] while applications to problem of pattern recognition and image processing, some of which involve the use of hierarchical data structures are given in [3].

By denoting $G + H$ the join of G and H a *wheel* W_n is defined as $W_n = K_1 + C_n$, for $n \geq 3$, a *fan* is $f_n = K_1 + P_n$ for $n \geq 1$ and *Jahangir graph* J_{2n}, ($n \geq 2$) (also known as *gear graph*) is obtained from the *wheel* W_{2n} by alternately deleting n spokes. Buczkowski et al. [2] determined the dimension of *wheel* W_n, Caceres et al. [7] the dimension of *fan* f_n and Tomescu and Javaid [18] the dimension of *Jahangir graph* J_{2n}.

Theorem 1 ([2,7,18]). *Let W_n be a wheel of order $n \geq 3$, f_n be fan of order $n \geq 1$ and J_{2n} be a Jahangir graph. Then*

(i) *For $n \geq 7$, $dim(W_n) = \lfloor \frac{2n+2}{5} \rfloor$;*
(ii) *For $n \geq 7$, $dim(f_n) = \lfloor \frac{2n+2}{5} \rfloor$;*
(iii) *For $n \geq 4$, $dim(J_{2n}) = \lfloor \frac{2n}{3} \rfloor$.*

The metric dimension of all these plane graphs depends upon the number of vertices in the graph.

On the other hand, we say that a family $\mathcal{G}$ of connected graphs is a family with constant metric dimension if $dim(G)$ is finite and does not depend upon the choice of G in $\mathcal{G}$. In [1] it was shown that a graph has metric dimension 1 if and only if it is a *path*, hence paths on n vertices constitute a family of graphs with constant metric dimension. Similarly, *cycles* with n (≥ 3) vertices also constitute such a family of graphs as their metric dimension is 2 and does not depend upon the number of vertices n [1]. In [6] it was proved that

$$dim(P_m \times C_n) = \begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{if } n \text{ is even.} \end{cases}$$

Since *prisms* D_n are the trivalent plane graphs obtained by the cross product of path P_2 with a cycle C_n, this implies that

$$dim(D_n) = \begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{if } n \text{ is even.} \end{cases}$$

So, prisms constitute a family of 3-*regular graphs* with constant metric dimension. Javaid et al. proved in [19] that the plane graph *antiprism* A_n constitute a family of regular graphs with constant metric dimension as $dim(A_n) = 3$ for every $n \geq 5$. The prism and the antiprism are *Archimedean* convex polytopes defined, e.g. in [12]. The metric dimension of cartesian product of graphs has been discussed in [6,15].

The metric dimension of some classes of *convex polytopes* has been studied in [10,11] recently where it was shown that these classes of convex polytopes have constant metric dimension.

Open Problem [11]: Let G' be the graph of a convex polytope obtained from the graph of convex polytope G by adding extra edges in G such that $V(G') = V(G)$. Is it the case that G' and G will always have the same metric dimension?

Note that the problem of determining whether $dim(G) < k$ is an *NP*-complete problem [9]. Some bounds for this invariant, in terms of the diameter of the graph, are given in [14] and it was shown in [1,3,14,15] that the metric dimension of trees can be determined efficiently. It appears unlikely that significant progress can be made in determining the dimension of a graph unless it belongs to a class for which the distances between vertices can be described in some systematic manner.

Bača defined in [20] the graph of convex polytope R_n which is obtained as a combination of the graph of a prism and the graph of an antiprism. The prism and antiprism have constant metric dimension [6,19] and it was proved in [10] that the graph of convex polytope R_n also has constant metric dimension.

In this paper, we extend this study to some classes of convex polytopes which are obtained by the combination of two different graphs of convex polytopes. We prove that these classes of convex polytopes have constant metric dimension and only three vertices appropriately chosen suffice to resolve all the vertices of these classes of convex polytopes. It is natural to ask for the characterization of classes of convex polytopes with constant metric dimension.

In what follows all indices i which do not satisfy the given inequalities will be taken modulo n.

2. The graph of convex polytope S_n

The graph of convex polytope S_n (Fig. 1) consists of $2n$ 3-sided faces, $2n$ 4-sided faces and a pair of n-sided faces, and is obtained by the combination of the graph of convex polytope R_n [20] and the graph of a prism D_n. We have

$$V(S_n) = \{a_i; b_i; c_i; d_i : 1 \leq i \leq n\}$$

and

$$E(S_n) = \{a_i a_{i+1}; b_i b_{i+1}; c_i c_{i+1}; d_i d_{i+1} : 1 \leq i \leq n\} \cup \{a_{i+1} b_i; a_i b_i; b_i c_i; c_i d_i : 1 \leq i \leq n\}.$$

The graph of convex polytope S_n can also be obtained from the graph of convex polytope Q_n defined in [20] by adding the edges $a_{i+1}b_i$; $c_i c_{i+1}$ and then deleting the edges $b_{i+1}c_i$. i.e., $V(S_n) = V(Q_n)$ and $E(S_n) = (E(Q_n) \cup \{a_{i+1}b_i; c_i c_{i+1} : 1 \leq i \leq n\}) \setminus \{b_{i+1}c_i : 1 \leq i \leq n\}$.

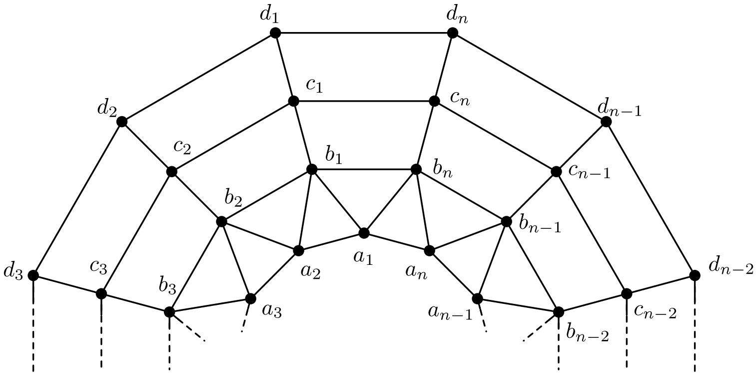

Fig. 1. The graph of convex polytope S_n.

For our purpose, we call the cycle induced by $\{a_i : 1 \le i \le n\}$, the inner cycle; cycle induced by $\{b_i : 1 \le i \le n\}$, the interior cycle; cycle induced by $\{c_i : 1 \le i \le n\}$, the exterior cycle and cycle induced by $\{d_i : 1 \le i \le n\}$, the outer cycle.

The metric dimension of the graph of convex polytope R_n and graph of a prism D_n have been studied in [6,10]. In the next theorem, we show that the metric dimension of the graph of convex polytope S_n is 3. Note that the choice of appropriate basis vertices (also referred to as landmarks in [13]) is core of the problem.

Theorem 2. *For $n \ge 6$, let the graph of convex polytopes be S_n, then $dim(S_n) = 3$;*

Proof. We will prove the above equality by double inequalities. We consider the two cases.

Case (i) When n is even.

In this case, we can write $n = 2k$, $k \ge 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_2, a_{k+1}\} \subset V(S_n)$, we show that W is a resolving set for S_n in this case. For this we give representations of any vertex of $V(S_n) \setminus W$ with respect to W.

Representations of the vertices on inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \le i \le k; \\ (2k-i+1, 2k-i+2, i-k-1), & k+2 \le i \le 2k. \end{cases}$$

Representations of the vertices on interior cycle are

$$r(b_i|W) = \begin{cases} (1, 1, k), & i = 1; \\ (i, i-1, k-i+1), & 2 \le i \le k; \\ (k, k, 1), & i = k+1; \\ (2k-i+1, 2k-i+2, i-k), & k+2 \le i \le 2k. \end{cases}$$

Representations of the vertices on exterior cycle are

$$r(c_i|W) = \begin{cases} (2, 2, k+1), & i = 1; \\ (i+1, i, k-i+2), & 2 \le i \le k; \\ (k+1, k+1, 2), & i = k+1; \\ (2k-i+2, 2k-i+3, i-k+1), & k+2 \le i \le 2k. \end{cases}$$

Representations of the vertices on outer cycle are

$$r(d_i|W) = \begin{cases} (3, 3, k+2), & i = 1; \\ (i+2, i+1, k-i+3), & 2 \le i \le k; \\ (k+2, k+2, 3), & i = k+1; \\ (2k-i+3, 2k-i+4, i-k+2), & k+2 \le i \le 2k. \end{cases}$$

We note that there are no two vertices having the same representations implying that $dim(S_n) \le 3$.

On the other hand, we show that $dim(S_n) \ge 3$ by proving that there is no resolving set W such that $|W| = 2$. Suppose on contrary that $dim(S_n) = 2$, then there are following possibilities to be discussed.

(1) Both vertices are in the inner cycle. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is a_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(a_n|\{a_1, a_t\}) = r(b_n|\{a_1, a_t\}) = (1, t)$ and for $t = k+1$, $r(a_2|\{a_1, a_{k+1}\}) = r(a_n|\{a_1, a_{k+1}\}) = (1, k-1)$, a contradiction.

(2) Both vertices are in the interior cycle. Without loss of generality we suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is b_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(a_1|\{b_1, b_t\}) = r(b_n|\{b_1, b_t\}) = (1, t)$ and for $t = k+1$, $r(b_2|\{b_1, b_{k+1}\}) = r(b_n|\{b_1, b_{k+1}\}) = (1, k-1)$, a contradiction.

(3) Both vertices are in the exterior cycle. Without loss of generality we suppose that one resolving vertex is c_1. Suppose that the second resolving vertex is c_t $(2 \leq t \leq k+1)$. Then for $2 \leq t \leq k+1$, we have $r(b_1|\{c_1, c_t\}) = r(d_1|\{c_1, c_t\}) = (1, t)$, a contradiction.

(4) Both vertices are in the outer cycle. Without loss of generality we suppose that one resolving vertex is d_1. Suppose that the second resolving vertex is d_t $(2 \leq t \leq k+1)$. Then for $2 \leq t \leq k$, we have $r(c_1|\{d_1, d_t\}) = r(d_n|\{d_1, d_t\}) = (1, t)$ and for $t = k+1$, $r(d_2|\{d_1, d_{k+1}\}) = r(d_n|\{d_1, d_{k+1}\}) = (1, k-1)$, a contradiction.

(5) One vertex is in the inner cycle and other in the interior cycle. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is b_t $(1 \leq t \leq k+1)$. Then for $1 \leq t \leq k-1$, we have $r(b_{n-1}|\{a_1, b_t\}) = r(c_n|\{a_1, b_t\}) = (2, t+1)$. If $t = k$, $r(a_n|\{a_1, b_k\}) = r(b_n|\{a_1, b_k\}) = (1, k)$ and when $t = k+1$, $r(a_2|\{a_1, b_{k+1}\}) = r(b_1|\{a_1, b_{k+1}\})$, a contradiction.

(6) One vertex is in the inner cycle and other in the exterior cycle. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is c_t $(1 \leq t \leq k+1)$. Then for $t = 1$, we have $r(a_2|\{a_1, c_1\}) = r(b_n|\{a_1, c_1\}) = (1, 2)$. If $2 \leq t \leq k+1$, $r(a_2|\{a_1, b_t\}) = r(b_1|\{a_1, b_t\}) = (1, t)$, a contradiction.

(7) One vertex is in the inner cycle and other in the outer cycle. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is d_t $(1 \leq t \leq k+1)$. Then for $t = 1$, we have $r(a_2|\{a_1, d_1\}) = r(b_n|\{a_1, d_1\}) = (1, 3)$. If $2 \leq t \leq k+1$, $r(a_2|\{a_1, d_t\}) = r(b_1|\{a_1, d_t\}) = (1, t+1)$, a contradiction.

(8) One vertex is in the interior cycle and other in the exterior cycle. Without loss of generality we suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is c_t $(1 \leq t \leq k+1)$. Then for $1 \leq t \leq k$, we have $r(a_1|\{b_1, c_t\}) = r(b_n|\{b_1, c_t\}) = (1, t+1)$ and when $t = k+1$, $r(a_1|\{b_1, c_{k+1}\}) = r(a_2|\{b_1, c_{k+1}\}) = (1, k+1)$, a contradiction.

(9) One vertex is in the interior cycle and other in the outer cycle. Without loss of generality we suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is d_t $(1 \leq t \leq k+1)$. Then for $1 \leq t \leq k$, we have $r(a_1|\{b_1, d_t\}) = r(b_n|\{b_1, d_t\}) = (1, t+2)$ and when $t = k+1$, $r(a_1|\{b_1, d_{k+1}\}) = r(a_2|\{b_1, d_{k+1}\}) = (1, k+2)$, a contradiction.

(10) One vertex is in the exterior cycle and other in the outer cycle. Without loss of generality we suppose that one resolving vertex is c_1. Suppose that the second resolving vertex is d_t $(1 \leq t \leq k+1)$. Then for $1 \leq t \leq k$, we have $r(a_1|\{c_1, d_t\}) = r(b_n|\{c_1, d_t\}) = (2, t+2)$ and when $t = k+1$, $r(a_1|\{c_1, d_{k+1}\}) = r(a_2|\{c_1, d_{k+1}\}) = (2, k+2)$, a contradiction.

Hence, from above it follows that there is no resolving set with two vertices for $V(S_n)$ implying that $dim(S_n) = 3$ in this case.

Case (ii) When n is odd.

In this case, we can write $n = 2k+1$, $k \geq 3$, $k \in \mathbf{Z}^+$. Again we show that $W = \{a_1, a_2, a_{k+1}\} \subset V(S_n)$ is a resolving set for S_n in this case. For this we give representations of any vertex of $V(S_n) \setminus W$ with respect to W.

Representations of the vertices on inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \leq i \leq k; \\ (k, k, 1), & i = k+2 \\ (2k-i+3, 2k-i+4, i-k-1), & k+3 \leq i \leq 2k+1. \end{cases}$$

Representations of the vertices on interior cycle are

$$r(b_i|W) = \begin{cases} (1, 1, k), & i = 1; \\ (i, i-1, k-i+1), & 2 \leq i \leq k; \\ (k+1, k, 1), & i = k+1; \\ (2k-i+2, 2k-i+3, i-k), & k+2 \leq i \leq 2k+1. \end{cases}$$

Representations of the vertices on exterior cycle are

$$r(c_i|W) = \begin{cases} (2, 2, k+1), & i = 1; \\ (i+1, i, k-i+2), & 2 \leq i \leq k; \\ (k+2, k+1, 2), & i = k+1; \\ (k+1, k+2, 3), & i = k+2; \\ (2k-i+3, 2k-i+4, i-k+1), & k+3 \leq i \leq 2k+1. \end{cases}$$

Representations of the vertices on outer cycle are

$$r(d_i|W) = \begin{cases} (3, 3, k+2), & i = 1; \\ (i+2, i+1, k-i+3), & 2 \leq i \leq k; \\ (k+3, k+2, 3), & i = k+1; \\ (2k-i+4, 2k-i+5, i-k+2), & k+2 \leq i \leq 2k+1. \end{cases}$$

Again we see that there are no two vertices having the same representations which implies that $dim(S_n) \leq 3$.

On the other hand, suppose that $dim(S_n) = 2$, then there are the same possibilities as in case (i) and contradiction can be deduced analogously. This implies that $dim(S_n) = 3$ in this case, which completes the proof. $\square$

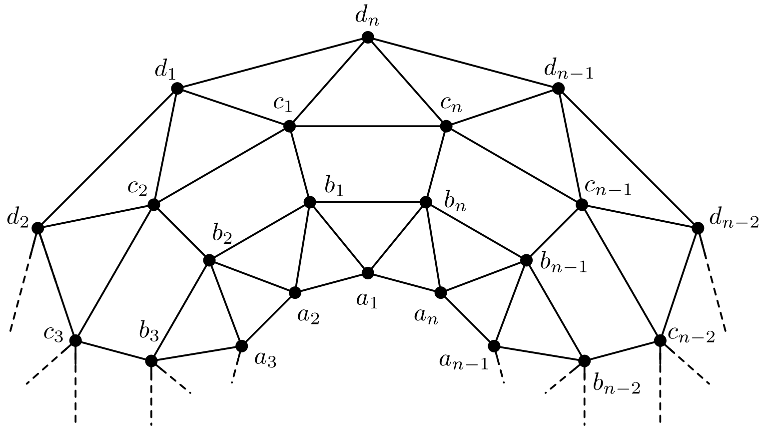

Fig. 2. The graph of convex polytope T_n.

3. The graph of convex polytope T_n

The graph of convex polytope T_n (Fig. 2) consists of $4n$ 3-sided faces, n 4-sided faces and a pair of n-sided faces, and is obtained by the combination of the graph of convex polytope R_n [20] and the graph of an antiprism A_n [21]. We have

$$V(T_n) = \{a_i; b_i; c_i; d_i : 1 \le i \le n\}$$

and

$$E(T_n) = \{a_ia_{i+1}; b_ib_{i+1}; c_ic_{i+1}; d_id_{i+1} : 1 \le i \le n\} \cup \{a_{i+1}b_i; a_ib_i; b_ic_i; c_id_i; c_{i+1}d_i : 1 \le i \le n\}.$$

The graph of convex polytope T_n can also be obtained from the graph of convex polytope Q_n defined in [20] by adding the edges $a_{i+1}b_i$; c_ic_{i+1}; $c_{i+1}d_i$ and then deleting the edges $b_{i+1}c_i$. i.e., $V(T_n) = V(Q_n)$ and $E(T_n) = (E(Q_n) \cup \{a_{i+1}b_i; c_ic_{i+1}; c_{i+1}d_i : 1 \le i \le n\}) \setminus \{b_{i+1}c_i : 1 \le i \le n\}$.

For our purpose, we call the cycle induced by $\{a_i : 1 \le i \le n\}$, the inner cycle; cycle induced by $\{b_i : 1 \le i \le n\}$, the interior cycle; cycle induced by $\{c_i : 1 \le i \le n\}$, the exterior cycle and cycle induced by $\{d_i : 1 \le i \le n\}$, the outer cycle.

The metric dimension of the graph of convex polytope R_n and graph of an antiprism A_n have been studied in [10,19]. In the next theorem, we show that the metric dimension of the graph of convex polytope T_n is 3. Again, choice of appropriate landmarks is crucial.

Theorem 3. *Let T_n denotes the graph of convex polytope; then $dim(T_n) = 3$ for every $n \ge 6$.*

Proof. We will prove the above equality by double inequalities. We consider the two cases.

Case (i) When n is even.

In this case, we can write $n = 2k$, $k \ge 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_2, a_{k+1}\} \subset V(T_n)$, we show that W is a resolving set for T_n in this case. For this we give representations of any vertex of $V(T_n) \setminus W$ with respect to W.

Representations of the vertices on inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \le i \le k; \\ (2k-i+2, 2k-i+3, i-k), & k+2 \le i \le 2k. \end{cases}$$

Representations of the vertices on interior cycle are

$$r(b_i|W) = \begin{cases} (1, 1, k), & i = 1; \\ (i, i-1, k-i+1), & 2 \le i \le k; \\ (k, k, 1), & i = k+1; \\ (2k-i+1, 2k-i+2, i-k), & k+2 \le i \le 2k. \end{cases}$$

Representations of the vertices on exterior cycle are

$$r(c_i|W) = \begin{cases} (2, 2, k+1), & i = 1; \\ (i+1, i, k-i+2), & 2 \le i \le k; \\ (k+1, k+1, 2), & i = k+1; \\ (2k-i+2, 2k-i+3, i-k+1), & k+2 \le i \le 2k. \end{cases}$$

Representations of the vertices on outer cycle are

$$r(d_i|W) = \begin{cases} (3, 3, k+1), & i = 1; \\ (i+2, i+1, k-i+2), & 2 \le i \le k-1; \\ (k+2, k+1, 3), & i = k; \\ (k+1, k+2, 3), & i = k+1; \\ (2k-i+2, 2k-i+3, i-k+2), & k+2 \le i \le 2k-1; \\ (3, 3, k+2), & i = 2k. \end{cases}$$

We note that there are no two vertices having the same representations implying that $dim(T_n) \le 3$.

On the other hand, we show that $dim(T_n) \ge 3$. Suppose on contrary that $dim(S_n) = 2$, then there are following possibilities to be discussed.

(1) Both vertices are in the inner cycle. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is a_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(a_n|\{a_1, a_t\}) = r(b_n|\{a_1, a_t\}) = (1, t)$ and for $t = k+1$, $r(a_2|\{a_1, a_{k+1}\}) = r(a_n|\{a_1, a_{k+1}\}) = (1, k-1)$, a contradiction.

(2) Both vertices are in the interior cycle. Without loss of generality we suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is b_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(a_1|\{b_1, b_t\}) = r(b_n|\{b_1, b_t\}) = (1, t)$ and for $t = k+1$, $r(b_2|\{b_1, b_{k+1}\}) = r(b_n|\{b_1, b_{k+1}\}) = (1, k-1)$, a contradiction.

(3) Both vertices are in the exterior cycle. Due to the symmetry of the graph, this case is analogous to case (2).

(4) Both vertices are in the outer cycle. This case is analogous to case (1).

(5) One vertex is in the inner cycle and other in the interior cycle. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is b_t $(1 \le t \le k+1)$. Then for $1 \le t \le k-1$, we have $r(b_{n-1}|\{a_1, b_t\}) = r(c_n|\{a_1, b_t\}) = (2, t+1)$. If $t = k$, $r(a_n|\{a_1, b_k\}) = r(b_n|\{a_1, b_k\}) = (1, k)$ and when $t = k+1$, $r(a_2|\{a_1, b_{k+1}\}) = r(b_1|\{a_1, b_{k+1}\})$, a contradiction.

(6) One vertex is in the inner cycle and other in the exterior cycle. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is c_t $(1 \le t \le k+1)$. Then for $t = 1$, we have $r(a_2|\{a_1, c_1\}) = r(b_n|\{a_1, c_1\}) = (1, 2)$. If $2 \le t \le k+1$, $r(a_2|\{a_1, b_t\}) = r(b_1|\{a_1, b_t\}) = (1, t)$, a contradiction.

(7) One vertex is in the inner cycle and other in the outer cycle. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is d_t $(1 \le t \le k+1)$. Then for $t = 1$, we have $r(a_2|\{a_1, d_1\}) = r(b_n|\{a_1, d_1\}) = (1, 3)$. If $2 \le t \le k$, $r(a_2|\{a_1, d_t\}) = r(b_1|\{a_1, d_t\}) = (1, t+1)$ and when $t = k+1$, $r(a_n|\{a_1, d_{k+1}\}) = r(b_n|\{a_1, d_{k+1}\}) = (1, t+1)$, a contradiction.

(8) One vertex is in the interior cycle and other in the exterior cycle. Without loss of generality we suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is c_t $(1 \le t \le k+1)$. Then for $1 \le t \le k$, we have $r(a_1|\{b_1, c_t\}) = r(b_n|\{b_1, c_t\}) = (1, t+1)$ and when $t = k+1$, $r(a_1|\{b_1, c_{k+1}\}) = r(a_2|\{b_1, c_{k+1}\}) = (1, k+1)$, a contradiction.

(9) One vertex is in the interior cycle and other in the outer cycle. This case is analogous to case (6).

(10) One vertex is in the exterior cycle and other in the outer cycle. This case is analogous to case (5) due to the symmetry of the graph.

Hence, from above it follows that there is no resolving set with two vertices for $V(T_n)$ implying that $dim(T_n) = 3$ in this case.

Case (ii) When n is odd.

In this case, we can write $n = 2k+1$, $k \ge 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_2, a_{k+1}\} \subset V(T_n)$, we show that W is a resolving set for T_n in this case. For this we give representations of any vertex of $V(T_n) \setminus W$ with respect to W.

Representations of the vertices on inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \le i \le k; \\ (k, k, 1), & i = k+2; \\ (2k-i+2, 2k-i+3, i-k-1), & k+3 \le i \le 2k+1. \end{cases}$$

Representations of the vertices on interior cycle are

$$r(b_i|W) = \begin{cases} (1, 1, k), & i = 1; \\ (i, i-1, k-i+1), & 2 \le i \le k; \\ (k+1, k+1, 1), & i = k+1; \\ (2k-i+2, 2k-i+3, i-k), & k+2 \le i \le 2k+1. \end{cases}$$

Representations of the vertices on exterior cycle are

$$r(c_i|W) = \begin{cases} (2, 2, k+1), & i = 1; \\ (i+1, i, k-i+2), & 2 \le i \le k; \\ (k+2, k+1, 2), & i = k+1; \\ (2k-i+3, 2k-i+4, i-k+1), & k+2 \le i \le 2k+1. \end{cases}$$

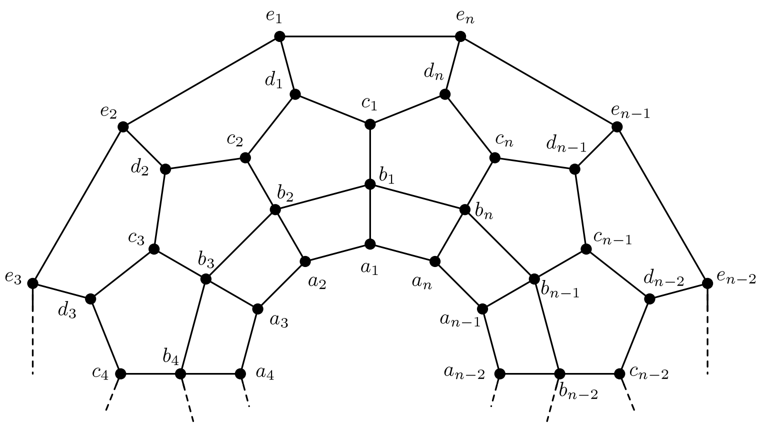

Fig. 3. The graph of convex polytope U_n.

Representations of the vertices on outer cycle are

$$r(d_i|W) = \begin{cases} (3, 3, k+1), & i = 1; \\ (i+2, i+1, k-i+2), & 2 \le i \le k-1; \\ (k+2, k+1, 3), & i = k; \\ (k+2, k+2, 3), & i = k+1; \\ (2k-i+3, 2k-i+4, i-k+2), & k+2 \le i \le 2k; \\ (3, 3, k+2), & i = 2k+1. \end{cases}$$

Again we see that there are no two vertices having the same representations which implies that $dim(T_n) \le 3$ in this case.

On the other hand, suppose that $dim(T_n) = 2$, then there are the same subcases as in case (i) and contradiction can be obtained analogously. This implies that $dim(T_n) = 3$ in this case, which completes the proof. □

4. The graph of convex polytope U_n

The graph of convex polytope U_n (Fig. 3) consists of n 4-sided faces, $2n$ 5-sided faces and a pair of n-sided faces, and is obtained as a combination of the graph of convex polytope $\mathbb{D}_n$ [21] and graph of a prism D_n. We have

$$V(U_n) = \{a_i; b_i; c_i; d_i; e_i : 1 \le i \le n\}$$

and

$$E(U_n) = \{a_ia_{i+1}; b_ib_{i+1}; e_ie_{i+1} : 1 \le i \le n\} \cup \{a_ib_i; b_ic_i; c_id_i; d_ie_i; c_{i+1}d_i : 1 \le i \le n\}.$$

For our purpose, we call the cycle induced by $\{a_i : 1 \le i \le n\}$, the inner cycle; cycle induced by $\{b_i : 1 \le i \le n\}$, the interior cycle; cycle induced by $\{c_i : 1 \le i \le n\} \cup \{d_i : 1 \le i \le n\}$, the exterior cycle and cycle induced by $\{e_i : 1 \le i \le n\}$, the outer cycle.

The metric dimension of the graph of convex polytope $\mathbb{D}_n$ and graph of a prism D_n have been studied in [6,10]. In the next theorem, we show that the metric dimension of the graph of convex polytope U_n is 3. Once again, the choice of appropriate landmarks is crucial.

Theorem 4. *Let U_n denotes the graph of convex polytope, then $dim(U_n) = 3$ for every $n \ge 6$;*

Proof. We will prove the above equality by double inequalities. We consider the two cases.

Case (i) When n is even.

In this case, we can write $n = 2k$, $k \ge 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_2, a_{k+1}\} \subset V(U_n)$, we show that W is a resolving set for U_n in this case. For this we give representations of any vertex of $V(U_n) \setminus W$ with respect to W.

Representations of the vertices on inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \le i \le k; \\ (2k-i+1, 2k-i+2, i-k-1), & k+2 \le i \le 2k. \end{cases}$$

Representations of the vertices on interior cycle are

$$r(b_i|W) = \begin{cases} (1, 2, k), & i = 1; \\ (i, i-1, k-i+2), & 2 \le i \le k+1; \\ (2k-i+2, 2k-i+3, i-k), & k+2 \le i \le 2k. \end{cases}$$

Representations of the vertices on exterior cycle are

$$r(c_i|W) = \begin{cases} (2, 3, k+2), & i = 1; \\ (i+1, i, k-i+3), & 2 \le i \le k; \\ (k+2, k+1, 3), & i = k+1; \\ (2k-i+3, 2k-i+4, i-k+1), & k+2 \le i \le 2k \end{cases}$$

and

$$r(d_i|W) = \begin{cases} (3, 3, k+3), & i = 1; \\ (i+2, i+1, k-i+3), & 2 \le i \le k; \\ (k+3, k+3, 3), & i = k+1; \\ (2k-i+3, 2k-i+4, i-k+2), & k+2 \le i \le 2k. \end{cases}$$

Representations of the vertices on outer cycle are

$$r(e_i|W) = \begin{cases} (4, 4, k+3), & i = 1; \\ (i+3, i+2, k-i+4), & 2 \le i \le k; \\ (k+3, k+3, 4), & i = k+1; \\ (2k-i+4, 2k-i+5, i-k+3), & k+2 \le i \le 2k. \end{cases}$$

We note that there are no two vertices having the same representations implying that $dim(U_n) \le 3$.

On the other hand, we show that $dim(U_n) \ge 3$. Suppose on contrary that $dim(V_n) = 2$, then there are following possibilities to be discussed.

(1) Both vertices are in the inner cycle. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is a_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(a_n|\{a_1, a_t\}) = r(b_1|\{a_1, a_t\}) = (1, t)$ and for $t = k+1$, $r(a_2|\{a_1, a_{k+1}\}) = r(a_n|\{a_1, a_{k+1}\}) = (1, k-1)$, a contradiction.

(2) Both vertices are in the interior cycle. Without loss of generality we suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is b_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(a_1|\{b_1, b_t\}) = r(b_n|\{b_1, b_t\}) = (1, t)$ and for $t = k+1$, $r(b_2|\{b_1, b_{k+1}\}) = r(b_n|\{b_1, b_{k+1}\}) = (1, k-1)$, a contradiction.

(3) Both vertices are in the exterior cycle. Here are the two subcases.

• Both vertices are in the set $\{c_i : 1 \le i \le n\}$. Without loss of generality we suppose that one resolving vertex is c_1. Suppose that the second resolving vertex is c_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(a_1|\{c_1, c_t\}) = r(b_n|\{c_1, c_t\}) = (2, t+1)$ and for $t = k+1$, $r(d_1|\{c_1, c_{k+1}\}) = r(d_n|\{c_1, c_{k+1}\}) = (1, k+3)$, a contradiction.

• Both vertices are in the set $\{d_i : 1 \le i \le n\}$. Without loss of generality we suppose that one resolving vertex is d_1. Suppose that the second resolving vertex is d_t $(2 \le t \le k+1)$. Then for $2 \le t \le k+1$, we have $r(b_1|\{d_1, d_t\}) = r(e_n|\{d_1, d_t\}) = (2, t+1)$, a contradiction.

• One vertex is in the set $\{c_i : 1 \le i \le n\}$ and other in the set $\{d_i : 1 \le i \le n\}$. Without loss of generality we suppose that one resolving vertex is c_1. Suppose that the second resolving vertex is d_t $(2 \le t \le k+1)$. Then for $t = 1$, we have $r(b_1|\{c_1, d_1\}) = r(d_n|\{c_1, d_1\}) = (1, 2)$. If $2 \le t \le k$, we have $r(a_2|\{c_1, d_t\}) = r(b_1|\{c_1, d_t\}) = (1, t+1)$ and for $t = k+1$, $r(a_n|\{c_1, d_{k+1}\}) = r(b_1|\{c_1, d_{k+1}\}) = (1, k+1)$, a contradiction.

(4) Both vertices are in the outer cycle. Without loss of generality we suppose that one resolving vertex is e_1. Suppose that the second resolving vertex is e_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(d_1|\{e_1, e_t\}) = r(e_n|\{e_1, e_t\}) = (1, t)$ and for $t = k+1$, $r(e_2|\{e_1, e_{k+1}\}) = r(e_n|\{e_1, e_{k+1}\}) = (1, k-1)$, a contradiction.

(5) One vertex is in the inner cycle and other in the interior cycle. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is b_t $(1 \le t \le k+1)$. Then for $1 \le t \le k$, we have $r(b_n|\{a_1, b_t\}) = r(c_1|\{a_1, b_t\}) = (2, t)$ and when $t = k+1$, $r(a_2|\{a_1, b_{k+1}\}) = r(a_n|\{a_1, b_{k+1}\}) = (1, k)$, a contradiction.

(6) One vertex is in the inner cycle and other in the exterior cycle. Here are the two subcases.

• One vertex is in the inner cycle and other in the set $\{c_i : 1 \le i \le n\}$. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is c_t $(1 \le t \le k+1)$. Then for $t = 1$, we have $r(a_2|\{a_1, c_1\}) = r(a_n|\{a_1, c_1\}) = (1, 3)$. If $2 \le t \le k+1$, $r(a_2|\{a_1, b_t\}) = r(b_1|\{a_1, b_t\}) = (1, t)$, a contradiction.

• One vertex is in the inner cycle and other in the set $\{d_i : 1 \le i \le n\}$. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is d_t $(1 \le t \le k+1)$. Then for $t = 1$, we have $r(d_2|\{a_1, d_1\}) = r(e_n|\{a_1, d_1\}) = (4, 2)$. If $2 \le t \le k$, $r(a_2|\{a_1, d_t\}) = r(b_1|\{a_1, d_t\}) = (1, t+1)$ and when $t = k+1$, $r(a_n|\{a_1, d_{k+1}\}) = r(b_1|\{a_1, d_{k+1}\}) = (1, k+1)$, a contradiction.

(7) One vertex is in the inner cycle and other in the outer cycle. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is e_t $(1 \le t \le k+1)$. Then for $t = 1$, we have $r(c_2|\{a_1, e_1\}) = r(d_n|\{a_1, e_1\}) = (1, 3)$. If $2 \le t \le k$, $r(a_2|\{a_1, e_t\}) = r(b_1|\{a_1, e_t\}) = (1, t+2)$ and when $t = k+1$, $r(a_n|\{a_1, e_{k+1}\}) = r(b_1|\{a_1, e_{k+1}\}) = (1, k+3)$, a contradiction.

(8) One vertex is in the interior cycle and other in the exterior cycle. Here are the two subcases.

• One vertex is in the interior cycle and other in the set $\{c_i : 1 \le i \le n\}$. Without loss of generality we suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is c_t $(1 \le t \le k+1)$. Then for $1 \le t \le k$, we have $r(a_1|\{b_1, c_t\}) = r(b_n|\{b_1, c_t\}) = (1, t+1)$ and when $t = k+1$, $r(b_2|\{b_1, c_{k+1}\}) = r(b_n|\{b_1, c_{k+1}\}) = (1, k)$, a contradiction.

• One vertex is in the interior cycle and other in the set $\{d_i : 1 \le i \le n\}$. Without loss of generality we suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is d_t $(1 \le t \le k+1)$. Then for $1 \le t \le k-1$, we have $r(a_1|\{b_1, d_t\}) = r(b_n|\{b_1, d_t\}) = (1, t+2)$. For $t = k, k+1$, $r(a_{n-2}|\{b_1, d_t\}) = r(e_n|\{b_1, d_t\}) = (3, t)$, a contradiction.

(9) One vertex is in the interior cycle and other in the outer cycle. Without loss of generality we suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is e_t $(1 \le t \le k+1)$. Then for $1 \le t \le k-1$, we have $r(a_1|\{b_1, e_t\}) = r(b_n|\{b_1, e_t\}) = (1, t+3)$. For $t = k$, $r(b_2|\{b_1, e_k\}) = r(c_1|\{b_1, e_k\}) = (1, k+1)$ and when $t = k+1$, $r(b_n|\{b_1, e_{k+1}\}) = r(c_1|\{b_1, e_{k+1}\}) = (1, k+1)$, a contradiction.

(10) One vertex is in the exterior cycle and other in the outer cycle. Here are the two subcases.

• One vertex is in the set $\{c_i : 1 \le i \le n\}$ and other in outer cycle. Due to the symmetry of the graph, this subcase is analogous to second subcase of case (8).

• One vertex is in the set $\{d_i : 1 \le i \le n\}$ and other in outer cycle. This subcase is analogous to first subcase of case (8).

Hence, from above it follows that there is no resolving set with two vertices for $V(U_n)$ implying that $dim(U_n) = 3$ in this case.

Case (ii) When n is odd.

In this case, we can write $n = 2k+1$, $k \ge 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_2, a_{k+1}\} \subset V(U_n)$, we show that W is a resolving set for U_n in this case. For this we give representations of any vertex of $V(U_n) \setminus W$ with respect to W.

Representations of the vertices on inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \le i \le k; \\ (k, k, 1), & i = k+2; \\ (2k-i+2, 2k-i+3, i-k-1), & k+3 \le i \le 2k+1. \end{cases}$$

Representations of the vertices on interior cycle are

$$r(b_i|W) = \begin{cases} (1, 2, k+1), & i = 1; \\ (i, i-1, k-i+2), & 2 \le i \le k+1; \\ (k+1, k+1, 2), & i = k+2; \\ (2k-i+3, 2k-i+4, i-k), & k+3 \le i \le 2k+1. \end{cases}$$

Representations of the vertices on exterior cycle are

$$r(c_i|W) = \begin{cases} (2, 3, k+2), & i = 1; \\ (i+1, i, k-i+3), & 2 \le i \le k+1; \\ (k+2, k+2, 3), & i = k+2; \\ (2k-i+4, 2k-i+5, i-k+1), & k+3 \le i \le 2k+1 \end{cases}$$

and

$$r(d_i|W) = \begin{cases} (3, 3, k+2), & i = 1; \\ (i+2, i+1, k-i+3), & 2 \le i \le k; \\ (k+3, k+2, 3), & i = k+1; \\ (k+2, k+1, 4), & i = k+2; \\ (k+1, k, 5), & i = k+2; \\ (2k-i+4, 2k-i+5, i-k+2), & k+3 \le i \le 2k+1. \end{cases}$$

Representations of the vertices on outer cycle are

$$r(e_i|W) = \begin{cases} (4, 4, k+3), & i = 1; \\ (i+3, i+2, k-i+4), & 2 \le i \le k; \\ (k+4, k+3, 4), & i = k+1; \\ (2k-i+5, 2k-i+6, i-k+3), & k+2 \le i \le 2k+1. \end{cases}$$

Again we see that there are no two vertices having the same representations which implies that $dim(U_n) \le 3$ in this case.

On the other hand, suppose that $dim(U_n) = 2$, then there are the same subcases as in case (i) and contradiction can be obtained analogously. This implies that $dim(U_n) = 3$ in this case, which completes the proof. □

5. Concluding remarks

In this paper, we have studied the metric dimension of some classes of convex polytopes which are obtained by the combination of two different convex polytopes. We have seen that the metric dimension of these classes of convex polytopes is finite and does not depend upon the number of vertices in these graphs and only three vertices appropriately chosen suffice to resolve all the vertices of these classes of convex polytopes. It is natural to ask for the characterization of classes of convex polytopes with constant metric dimension.

Note that in [3] Melter and Tomescu gave an example of infinite regular plane graph (namely the digital plane endowed with city-block distance) having no finite metric basis. We close this section by raising a question as an open problem that naturally arises from the text.

Open Problem: Let G be the graph of a convex polytope which is obtained by joining the graph of two different convex polytopes G_1 and G_2 (such that the outer cycle of G_1 is the inner cycle of G_2) both having constant metric dimension. Is it the case that G will always have the constant metric dimension?

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